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Radiation reaction in classical and quantum electrodynamics

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References

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Radiation by accelerated charges

- Accelerated electric charges, an electron (charge e and mass m) for definiteness, emit electromagnetic radiation (units with }=c=1)
- Non-relativistically the energy emitted per unit time is given by the Larmor formula

$$\frac{d\mathcal{E}}{dt} = \frac{2}{3} e^2 \dot{\boldsymbol{v}}^2$$

- The corresponding relativistic formula reads

$$\frac{d\mathcal{E}}{dt} = -\frac{2}{3} \frac{e^2}{m^2} \frac{dp^\mu}{ds} \frac{dp_\mu}{ds}$$

  and shows the invariance of the emitted power (s is the electron's proper time)
- The exact dynamics of the electron in an external field includes the effects of this energy loss (and of the related momentum and angular momentum losses)

Radiation reaction in CED

- How can we include the energy-momentum loss due to radiation into the equation of motion of the electron?
- Non-relativistically, if the electron experiences a force **F**, we write its equation of motion as

$$m\dot{\boldsymbol{v}} = \boldsymbol{F} + \boldsymbol{F}_{\mathrm{rad}}$$

where $\mathbf{F}_{\mathrm{rad}}$ is the force responsible of the electromagnetic energy loss:

$$\int_{t_1}^{t_2} dt\, \boldsymbol{F}_{\mathrm{rad}} \cdot \boldsymbol{v} = -\frac{2}{3}e^2 \int_{t_1}^{t_2} dt\, \dot{\boldsymbol{v}}^2 = -\frac{2}{3}e^2 (\boldsymbol{v}\cdot\dot{\boldsymbol{v}})|_{t_1}^{t_2} + \frac{2}{3}e^2 \int_{t_1}^{t_2} dt\, \ddot{\boldsymbol{v}}\cdot\boldsymbol{v}$$

- If the motion is periodic or such that $\mathbf{v}$¢$\dot{\mathbf{v}}$=0 at t_1 and t_2, one can identify

$$\boldsymbol{F}_{\mathrm{rad}} = \frac{2}{3}e^2\ddot{\boldsymbol{v}}$$

- The radiation force depends on the derivative of the acceleration of the electron

- There are several relativistic approaches to radiation reaction
- One has to solve self consistently the coupled Lorentz and Maxwell equations (Barut 1980)

$$\begin{aligned} m_0 \frac{du^\mu}{ds} &= eF_T^{\mu\nu} u_\nu \\ \partial_\mu F_T^{\mu\nu} &= 4\pi e \int ds \delta(x - x(s)) u^\nu \end{aligned} \quad \xrightarrow{\text{Lorenz gauge}} \quad \begin{aligned} m_0 \frac{du^\mu}{ds} &= e(\partial^\mu A_T^\nu - \partial^\nu A_T^\mu) u_\nu \\ \Box A_T^\nu &= 4\pi e \int ds \delta(x - x(s)) u^\nu \end{aligned}$$

where now m_0 is the electron's bare mass and $F_{T,10}$=@$_1A_{T,o}${@$_oA_{T,1}$ is the total electromagnetic ¯eld (external ¯eld plus the one generated by the electron)

- One ¯rst solves the inhomogeneous wave equation exactly with the Green's-function method

$$\Box A_T^\nu = 4\pi e \int ds \delta(x - x(s)) u^\nu = 4\pi j^\nu(x) \longrightarrow A_T^\nu(x) = A^\nu(x) + \int d^4x' \mathcal{D}_R(x - x') j^\nu(x')$$

and then re-substitute the solution into the Lorentz equation:

$$(m_0 + \delta m) \frac{du^\mu}{ds} = eF^{\mu\nu} u_\nu + \frac{2}{3} e^2 \left(\frac{d^2u^\mu}{ds^2} + \frac{du^\nu}{ds} \frac{du_\nu}{ds} u^\mu \right)$$

where ±m is a quantity which diverges for a pointlike charge

- After "classical mass renormalization" one obtains the Lorentz-Abraham-Dirac (LAD) equation

$$e^{-s/\tau}\, m\frac{du^\mu}{ds} = \left[eF^{\mu\nu}u_\nu + \frac{2}{3}e^2\left(\frac{d^2u^\mu}{ds^2} + \frac{du^\nu}{ds}\frac{du_\nu}{ds}u^\mu\right)\right] e^{-s/\tau}$$

- The LAD equation is plagued by serious inconsistencies: runaway solutions. Consider its three-dimensional non-relativistic limit

$$m\frac{d\boldsymbol{v}}{dt} = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) + \frac{2}{3}e^2\frac{d^2\boldsymbol{v}}{dt^2}$$

  Even in the free case **E**=**B**=**0**, it admits the solution **a**(t)=$\mathbf{a}_0$e$^{t/¿}$, where ¿=(2/3)e^2/m»10$^{\{24}$ s

- ¿=(2/3)r$_0$, with r$_0$=2.8£10$^{\{13}$ cm being the classical electron radius
- Avoiding the runaways: integro-differential LAD equation

$$m\frac{du^\mu}{ds} = \frac{e^{s/\tau}}{\tau}\int_s^\infty ds' e^{-s'/\tau}\left[eF^{\mu\nu}u_\nu + \frac{2}{3}e^2\frac{du^\nu}{ds'}\frac{du_\nu}{ds'}u^\mu\right]$$

- Problem: preacceleration at time scales of the order of ¿
- The classical time scale of ¿ is about two orders of magnitude smaller than the typical quantum scale ¸$_C$/c=}/mc^2= 1.3£10$^{\{21}$ s (the constant of proportionality is the fine-structure constant ®=e^2/}c¼1/137)

Field scale	Critical fields of CED: $E_0=mc^2/|e|r_0=1.8\times10^{18}$ V/cm $B_0=mc^2/|e|r_0=6.0\times10^{15}$ G	Critical fields of QED: $E_{cr}=mc^2/|e|\lambda_C=1.3\times10^{16}$ V/cm $B_{cr}=mc^2/|e|\lambda_C=4.4\times10^{13}$ G
Intensity scale	$I_0=cE_0^2/4\pi=8.6\times10^{33}$ W/cm^2	$I_{cr}=cE_{cr}^2/4\pi=4.6\times10^{29}$ W/cm^2

- If $\max|F(x)|_{IRF}\ll F_0=(E_0,B_0)$ and $\max|F^{-1}(x)\partial F(x)/\partial x|_{IRF}\ll 1/r_0$, one can replace the four-acceleration du^μ/ds in the radiation-reaction force in the LAD equation

$$m\frac{du^\mu}{ds}=eF^{\mu\nu}u_\nu+\boxed{\frac{2}{3}e^2\left(\frac{d^2u^\mu}{ds^2}+\frac{du^\nu}{ds}\frac{du_\nu}{ds}u^\mu\right)}$$

with the zero-order four-acceleration $eF^{\mu\nu}u_\nu/m$ (Landau and Lifshitz 1947)

- Since $(E_0,B_0)=(E_{cr},B_{cr})/\alpha\approx137(E_{cr},B_{cr})$ and $r_0=\alpha\lambda_C\approx\lambda_C/137$, the above conditions are always fulfilled in the realm of CED

- The resulting equation

$$m\frac{du^\mu}{ds}=eF^{\mu\nu}u_\nu+\frac{2}{3}e^2\left[\frac{e}{m}(\partial_\alpha F^{\mu\nu})u^\alpha u_\nu-\frac{e^2}{m^2}F^{\mu\nu}F_{\alpha\nu}u^\alpha+\frac{e^2}{m^2}(F^{\alpha\nu}u_\nu)(F_{\alpha\lambda}u^\lambda)u^\mu\right]$$

is known as Landau-Lifshitz (LL) equation and it has been recently tested experimentally (see talks by J. Cole and G. Sarri)

- Two important remarks:
  1. The LL equation is safe from inconsistencies and it includes all physical solutions of the LAD equation (Spohn 2001)
  2. The LL equation can be directly derived from QED (Krivitsky and Tsytovich 1991)
- The LAD equation is "too exact" (but in a wrong way):

$$m\frac{du^\mu}{ds} = a_1^\mu e + a_2^\mu e^2 + a_3^\mu e^3 + a_4^\mu e^4 + \cdots$$

- In the LAD equation the series in e is "summed" exactly (essential non-perturbative effects in e are predicted)
- Lower-order terms in } are much larger than higher-order terms like $a_4^\mu e^4$ (see Ilderton and Torgrimsson 2013)

- In the ultrarelativistic case radiation-reaction e®ects
  1. are mainly due to the “Larmor” damping term

$$m\frac{du^\mu}{ds} = eF^{\mu\nu}u_\nu + \frac{2}{3}e^2\left[\frac{e}{m}(\partial_\alpha F^{\mu\nu})u^\alpha u_\nu - \frac{e^2}{m^2}F^{\mu\nu}F_{\alpha\nu}u^\alpha + \frac{e^2}{m^2}(F^{\alpha\nu}u_\nu)(F_{\alpha\lambda}u^\lambda)u^\mu\right]$$

  2. scale with the parameter R_C©, where © is the total phase of the laser pulse and

$$R_C = \alpha\frac{\omega_0}{m}\gamma_0(1+\beta_0)\xi^2 = \alpha\chi\xi$$

- The condition R_Ct1 means that the energy emitted by the electron in one laser period is of the order of the initial energy (classical radiation dominated regime) (Landau and Lifshitz 1975, Koga et al., Phys. Plasmas 2005, ADP 2008)

LAD, LL, and FOC equation

- Starting from the coupled Lorentz and Maxwell's equations, one obtains the LAD equation

$$m\frac{du^\mu}{ds} = eF^{\mu\nu}u_\nu + \frac{2}{3}e^2\left(\frac{d^2u^\mu}{ds^2} + \frac{du^\nu}{ds}\frac{du_\nu}{ds}u^\mu\right)$$

- By carrying out the replacement $du^{1}/ds!eF^{1o}u_{o}/m$, one obtains the LL equation

$$m\frac{du^\mu}{ds} = eF^{\mu\nu}u_\nu + \frac{2}{3}e^2\left[\frac{e}{m}(\partial_\alpha F^{\mu\nu})u^\alpha u_\nu - \frac{e^2}{m^2}F^{\mu\nu}F_{\alpha\nu}u^\alpha + \frac{e^2}{m^2}(F^{\alpha\nu}u_\nu)(F_{\alpha\lambda}u^\lambda)u^\mu\right]$$

- The replacement has been carried out twice in the Schott term:

$$\frac{d^2u^\mu}{ds^2} \to \frac{d}{ds}\left(\frac{e}{m}F^{\mu\nu}u_\nu\right) = \frac{e}{m}\partial_\alpha F^{\mu\nu}u^\alpha u_\nu + \frac{e}{m}F^{\mu\nu}\frac{du_\nu}{ds} \to \frac{e}{m}\partial_\alpha F^{\mu\nu}u^\alpha u_\nu + \frac{e^2}{m^2}F^{\mu\nu}F_{\nu\alpha}u^\alpha$$

- By carrying out the replacement once one obtains the relativistic Ford-O'Connell (FOC) equation

$$m\frac{du^\mu}{ds} = eF^{\mu\nu}u_\nu + \frac{2}{3}\frac{e^3}{m}\left[\frac{d}{ds}(F^{\mu\nu}u_\nu) + \frac{du^\nu}{ds}F_{\nu\alpha}u^\alpha u^\mu\right]$$

- The replacement in the Larmor term is carried out once in order to conserve the on-shell condition

- Within classical electrodynamics the LAD, LL, and FOC equations are equivalent (see also Kravets et al. 2013)
- The three equations conserve the on-shell condition
- By integrating the LAD equation with respect to s, one obtains

$$mu^\mu(+\infty) - mu^\mu(-\infty) = e\int_{-\infty}^{+\infty} ds\, F^{\mu\nu}u_\nu + \frac{2}{3}e^2\int_{-\infty}^{+\infty} ds\, \frac{du^\nu}{ds}\frac{du_\nu}{ds}u^\mu$$

  - Assumption: $du^{\prime}(+1)/ds= du^{\prime}(\{1)/ds =0$
- By integrating the LL equation with respect to s, one obtains

$$mu^\mu(+\infty) - mu^\mu(-\infty) = e\int_{-\infty}^{+\infty} ds\, F^{\mu\nu}u_\nu + \frac{2}{3}e^2\int_{-\infty}^{+\infty} ds\left[\frac{e}{m}(\partial_\alpha F^{\mu\nu})u^\alpha u_\nu - \frac{e^2}{m^2}F^{\mu\nu}F_{\alpha\nu}u^\alpha\right] + \frac{2}{3}\frac{e^4}{m^2}\int_{-\infty}^{+\infty} ds\,(F^{\alpha\nu}u_\nu)(F_{\alpha\lambda}u^\lambda)u^\mu$$

  - The Larmor term is the integral of the classical limit $(2/3)e^2m^2\hat{A}^2(s)$ of the quantum intensity of radiation
- By integrating the FOC equation with respect to s, one obtains

$$mu^\mu(+\infty) - mu^\mu(-\infty) = e\int_{-\infty}^{+\infty} ds\, F^{\mu\nu}u_\nu + \frac{2}{3}\frac{e^3}{m}\int_{-\infty}^{+\infty} ds\, \frac{du^\nu}{ds}F_{\nu\alpha}u^\alpha u^\mu$$

  - Assumption (reasonable): $F^{1o}(+1) = F^{1o}(\{1) =0$

Radiation reaction in QED

- We introduced the problem of radiation reaction in CED by saying that the Lorentz equation has to be modified as it does account for the energy-momentum loss of the accelerating and then emitting electron
- Thus one could be tempted to say that radiation reaction is automatically taken into account in QED already in the "basic" emission process (nonlinear single Compton scattering)

because photon recoil, i.e., the energy-momentum subtracted by the photon to the electron is automatically included

- However, this cannot be the case because

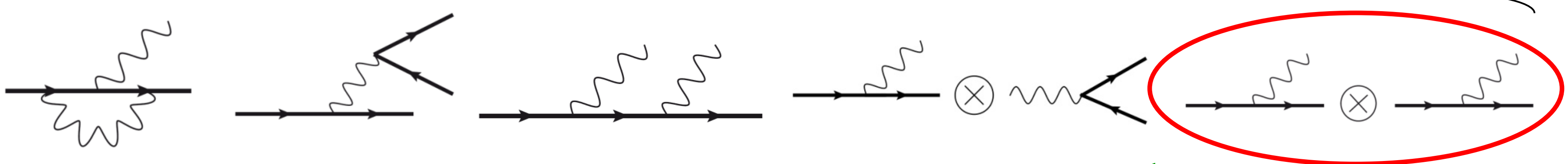
1. in the classical limit Â¿1, the spectrum of nonlinear Compton scattering goes into the classical spectrum calculated via the Lorentz equation, i.e., without radiation reaction
2. the photon recoil }! is proportional to } and it does not have a classical analogue
3. radiation reaction would always be a small correction classically, which is not the case in the radiation dominated regime

- To determine the dynamics of the electron via the Lorentz-Abraham-Dirac equation amounts to solve self-consistently Maxwell and Lorentz equations

$$m_0\frac{du^\mu}{ds} = eF_T^{\mu\nu}u_\nu \qquad \partial_\mu F_T^{\mu\nu} = e\int ds\,\delta(x-x(s))u^\nu \quad \longleftrightarrow \quad m\frac{du^\mu}{ds} = eF^{\mu\nu}u_\nu + \frac{2}{3}e^2\left(\frac{d^2u^\mu}{ds^2} + \frac{du^\nu}{ds}\frac{du_\nu}{ds}u^\mu\right)$$

- This corresponds in QED to determine the evolution of a single-electron state in background field+radiation ¯eld generated by the electron

$|t=-\infty\rangle = |e^-\rangle \longrightarrow$ Complete evolution operator (S-matrix) $\longrightarrow |t=+\infty\rangle$

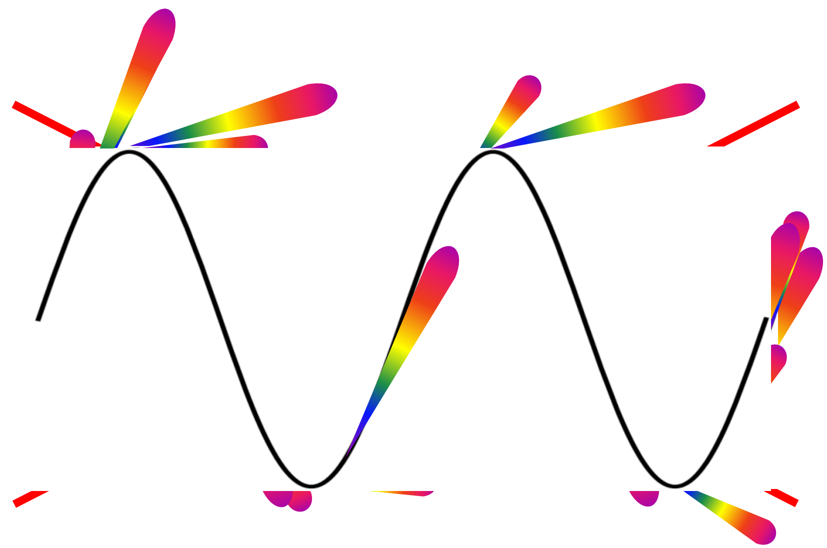


- In strong-field QED including "radiation reaction" amounts in accounting for all possible processes arising with an electron in the initial state
- At » À 1 and Â . 1 (moderately quantum regime) the multiple incoherent emission gives the main contribution (ADP et al. 2010). This starts playing a role if the total probability P_1 »®»© (Nikishov and Ritus 1964) of emitting one photon in a laser pulse exceeds unity

- Quantum analogous of each term in the LAD equation

Lorentz-Abraham-Dirac equation	$m\frac{du^\mu}{ds} = \boxed{eF^{\mu\nu}u_\nu} + \boxed{\frac{2}{3}e^2\left(\frac{d^2u^\mu}{ds^2}\right.} \boxed{\left.+\frac{du^\nu}{ds}\frac{du_\nu}{ds}u^\mu\right)}$

- The Larmor term corresponds to the cascade emission of many photons (Elkina et al. 2011)
- No multiple coherent emission
- Classical limit: the electron emits a large number of photons (N»®»©!1) but all with a small recoil (!»ÂE$_0$!0), in such a way that the average energy emitted (N!»®» Â©E$_0$=R$_C$©E$_0$) is finite



- Quantum radiation dominated regime (ADP et al. 2010): multiple photon emission already in one laser period (emission probability in one laser period P$_1$ »®» & 1) with a large recoil (Â . 1)
- The Schott term corresponds to radiative corrections and is usually negligible for high-intensity lasers in the ultrarelativistic regime

When does radiation reaction become important?

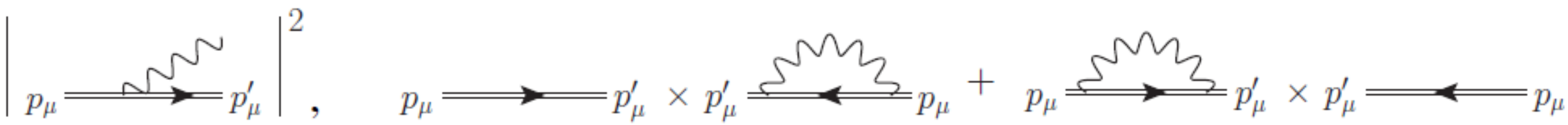
	CED	QED
Physical condition	When the total energy emitted is of the same order of the initial electron energy.	When the total probability P_1 of emitting one photon is larger than unity (it indicates that incoherent multiphoton emission occurs). Remind that P_1 »®»© (Ritus 1985).
Mathematical condition	®Â»©& 1	®»©& 1

Classical and quantum radiation dominated regime

	CED	QED
Radiation reaction parameter	R_C=®Â»	R_Q=®»
Physical meaning	Energy emitted in one laser period in units of the initial electron energy	Average number of photons emitted incoherently in one laser period
Radiation dominated regime	Â¿ 1 and R_C=®Â» & 1	Â . 1 and R_Q=®» & 1

One-particle approach to radiation reaction in QED

- Classically radiation reaction effects primarily alter the variation of the electron momentum with time (Lorentz vs LL equation)
- A convenient way especially to understand the classical limit of classical radiation reaction is to investigate the average momentum of a single electron driven by an external field (Ilderton and Torgrimsson 2013)
- Non-perturbative calculations in a plane wave are performed within the so-called light-cone quantization in the Furry picture
- At the leading order the diagrams



contribute to radiation reaction

- By calculating the first-order correction to the average electron momentum within QED and the classical limit, it has been shown that among the proposed classical equations only the LL, LAD (and the Ford-O'Connell) equations are compatible with the quantum theory

Kinetic approach to quantum radiation reaction

- If the electron emits sequentially a large number of photons, a kinetic approach is suitable to treat the problem
- Setup: an electron bunch head-on collides with a plane wave

- Parameters regime:
  1. the electron bunch is ultra-relativistic and it is barely deviated by the laser field from its initial direction of propagation: **"*Àm» Àm** (Landau and Lifshitz 1975)
  2. Quantum effects are “moderately” important: Â*=(2"*/m) (E_L /E_{cr}) . 1 (Ritus 1985)
- Corresponding simplifying assumptions:
  1. the one-dimensional kinetic approach can be employed
  2. electron-positron pair production can be neglected
- It is convenient to employ the coordinates: Á=t{y, T=(t+y)/2 and **r**$_?$=(x,z), and the corresponding momenta components P=("+p_y)/2, $p_{\{}$="{p_y and **p**$_?$=(p_x, p_z), as the field depends only on Á and the quantities $p_{\{}$ and **p**$_?$ are constant of motions

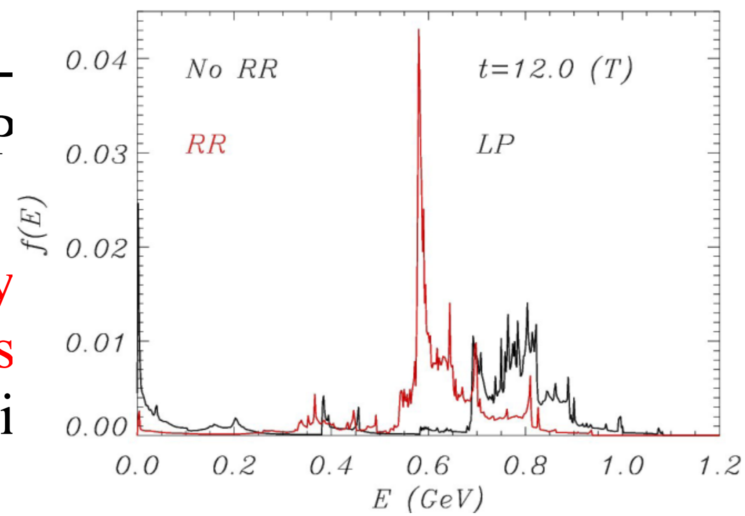
- The kinetic equations in our regime are (Baier et al. 1998)

$$\boxed{\frac{\partial n_e(\varphi,p_-)}{\partial\varphi}=\int_{p_-}^{\infty}dp_{i,-}\,n_e(\varphi,p_{i,-})\frac{dP_{p_{i,-}}}{d\varphi dp_-}-n_e(\varphi,p_-)\int_0^{p_-}dk_-\frac{dP_{p_-}}{d\varphi dk_-}}$$

$$\frac{\partial n_\gamma(\varphi,k_-)}{\partial\varphi}=\int_{k_-}^{\infty}dp_{i,-}\,n_e(\varphi,p_{i,-})\frac{dP_{p_{i,-}}}{d\varphi dk_-},$$

where $n_{e/\circ}$(',p$_{\{}$)=electron/photon distribution function, '=!$_0$Á=laser phase

- The two equations are not coupled and we consider only the first
- Motivation: radiation reaction in classical electrodynamics acts as a beneficial cooling mechanism
- Example: energy spectrum of a laser-generated ion beam (Tamburini et al. NJP 2010)
- Cooling mechanism: high-energy particles emit more than low-energy ones and the phase space contracts (Tamburini et al. NIMA 2011)



- What happens when QED effects set in (Neitz and ADP 2013)?

- The classical-quantum transition can be studied by expanding the kinetic equation at small values of the quantum parameter Â(',p{)

Order Â²(',p{): Liouville-like deterministic equation

$$\frac{\partial n_e}{\partial \varphi} = -\frac{\partial}{\partial p_-}\left(n_e \frac{dp_-}{d\varphi}\right), \quad \frac{dp_-}{d\varphi} = -\frac{I_{cl}(\varphi, p_-)}{\omega_0} = -\frac{2}{3}\alpha\frac{m^2}{\omega_0}\chi^2(\varphi, p_-)$$

corresponding to the LL dynamics (classical radiation reaction)

Order Â³(',p{): Fokker-Planck-like diffusion equation

$$\frac{\partial n_e}{\partial \varphi} = -\frac{\partial}{\partial p_-}[A(\varphi, p_-) n_e] + \frac{1}{2}\frac{\partial^2}{\partial p_-^2}[B(\varphi, p_-) n_e]$$

$$A(\varphi, p_-) = -\frac{I_{cl}(\varphi, p_-)}{\omega_0}\left[1 - \frac{55\sqrt{3}}{16}\chi(\varphi, p_-)\right]$$

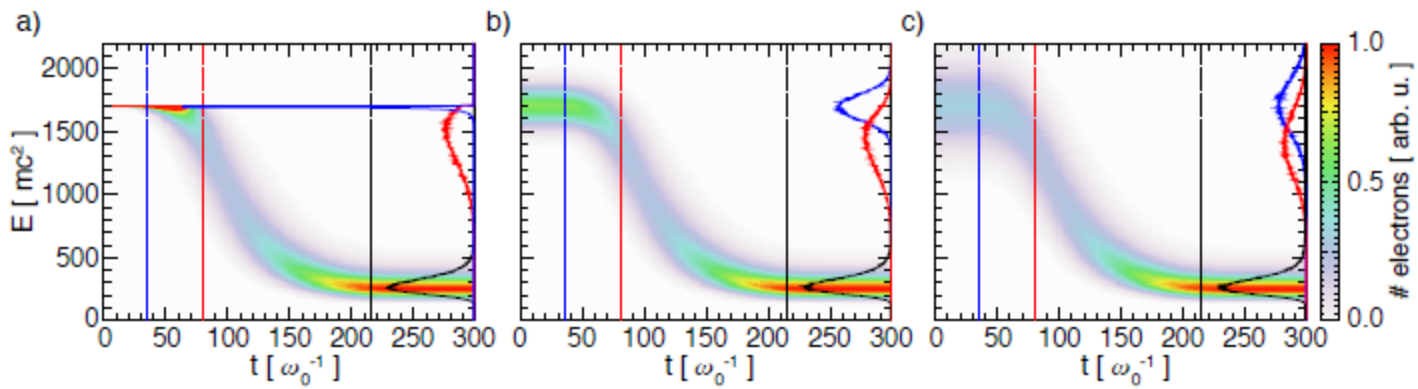
$$B(\varphi, p_-) = \frac{\alpha m^2}{3\omega_0}\frac{55}{8\sqrt{3}}p_-\chi^3(\varphi, p_-)$$

- Quantum effects induce:
  1. a correction to the intensity of radiation in agreement with the expansion of the corresponding quantum intensity of radiation I_q(',p{) (Ritus 1985)
  2. the appearance of the diffusion term
- The diffusion term is related to the stochasticity of quantum radiation reaction. The Fokker-Planck equation can be related to the single-particle stochastic equation (Gardiner 2009)

$$dp_- = -A(\varphi, p_-)d\varphi + \sqrt{B(\varphi, p_-)}dW$$

with dW being an infinitesimal stochastic function

- In general one obtains that quantum diffusion terms tend to broaden the electron energy distribution
- This is not a general result because the more the electron beam becomes classical by loosing energy, the more classical features become dominant
- For long pulses the electron energy distribution may initially narrow and then broaden again (Vranic et al. 2016)



- This conclusion has been confirmed by means of an analysis of the momenta of the electron energy distribution valid at arbitrary values of the quantum nonlinearity parameters (Niel et al. 2017)

Conclusions

- Radiation reaction is one of the oldest problems in electrodynamics and it is so fundamental that it has implications in various areas of physics:
  - astrophysics (motion of electrons and positrons around magnetized neutron stars or during supernova explosions)
  - fundamental physics (mass renormalization and quantum origin of radiation reaction)
  - accelerator physics (precise determination of electron trajectory, for example, in synchrotrons)
- Intense electromagnetic fields are required to make radiation-reaction effects sufficiently large to be measurable
- Already available laser and electron beam technology allows to test experimentally the equations underlying radiation reaction both in the classical and in the quantum regime